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Topological currents in relativistic Schrödinger theory

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Abstract. In relativistic Schrödinger theory, additional conservation laws arise of topological origin. These are due to the existence of *topological currents* which are built up by the exclusive use of operators, whereas the *matter currents* are composed of the densities. The general concepts and results are exemplified by considering a specific (Dirac) spinor field over the Robertson–Walker universes. The invariant, associated to the topological current, can be explicitly determined for SU(2)-bundles.

1. Introduction

In any field theory of matter, quantum or classical, the conservation laws play a dominant role. These conservation laws are frequently reformulated in differential terms as some kind of continuity equation for the corresponding density (for example current density j_μ or energy–momentum density $T_{\mu\nu}$, etc). Here the general belief is that the scientific aesthetics and internal consistency of a theory should be estimated by looking at the way in which the conservation laws are built into the basic structure of that theory. The two famous yardsticks in this respect are Einsteins gravitation theory and the Yang–Mills equations. Let us first have a glance at these two successful theories before making the aim of the present paper more precise.

Remember that Einstein's equations for determining the metric $g_{\mu\nu}$ of pseudo-Riemannian spacetime read†

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi \frac{L_p^2}{\hbar c} T_{\mu\nu}. \tag{1.1}$$

On the other hand, for any Riemannian curvature tensor $R_{\mu\nu\lambda\sigma}$ one has the Bianchi identity

$$\nabla_\rho R_{\mu\nu\lambda\sigma} + \nabla_\lambda R_{\mu\nu\sigma\rho} + \nabla_\sigma R_{\mu\nu\rho\lambda} \equiv 0. \tag{1.2}$$

Thus, contract this identity twice and find

$$\begin{aligned} &\nabla^\mu (R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}) \equiv 0 \\ &(R_{\mu\nu} \doteqdot R^\lambda_{\mu\lambda\nu},\, R \doteqdot R^\mu_\mu). \end{aligned} \tag{1.3}$$

But applying this result to Einstein's equations (1.1) readily yields the energy–momentum conservation law in the form

$$\nabla^\mu T_{\mu\nu} \equiv 0. \tag{1.4}$$

† Throughout the paper we work over a pseudo-Riemannian spacetime as base manifold with the (coordinate-) covariant derivative ∇ referring to the Levi–Civita connection of the tangent metric $g_{\mu\nu}$.

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In this way, the validity of the physically significant conservation law (1.4) has been traced back to Einstein's fortunate choice of the equation of motion (1.1) with regard to the mathematical identity (1.2).

The same successful procedure of combining the identities (of the underlying mathematical structure) with the dynamical equations in order to generate the conservation laws is also observed for the Yang–Mills theory. Here, the sources of the 'field strength' $\mathcal{F}_{\mu\nu}$ are identified by the field equations with the matter currents $\mathcal{J}_\nu$ through

$$\begin{gathered}\mathcal{D}^\mu \mathcal{F}_{\mu\nu} = 4\pi \mathcal{J}_\nu \\ (\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} \doteqdot \nabla_\lambda \mathcal{F}_{\mu\nu} + [\mathcal{A}_\lambda, \mathcal{F}_{\mu\nu}], \qquad \bar{\mathcal{A}}_\lambda = -\mathcal{A}_\lambda).\end{gathered} \tag{1.5}$$

On the other hand, for the curvature $\mathcal{F}_{\mu\nu}$ of the corresponding fibre bundle we have the identity

$$[\mathcal{D}_\lambda \mathcal{D}_\sigma - \mathcal{D}_\sigma \mathcal{D}_\lambda]\mathcal{F}_{\mu\nu} \equiv [\mathcal{F}_{\lambda\sigma}, \mathcal{F}_{\mu\nu}] - R^\rho_{\mu\lambda\sigma}\mathcal{F}_{\rho\nu} - R^\rho_{\nu\lambda\sigma}\mathcal{F}_{\mu\rho}. \tag{1.6}$$

Thus, contract again this identity twice and find

$$\mathcal{D}^\mu \mathcal{D}^\nu \mathcal{F}_{\mu\nu} \equiv 0. \tag{1.7}$$

But applying this result to the Yang–Mills equations (1.5) readily yields the continuity equation for the current density $\mathcal{J}_\nu$

$$\mathcal{D}^\mu \mathcal{J}_\mu \equiv 0. \tag{1.8}$$

(For ordinary Maxwellian electrodynamics this is the charge conservation law: $\nabla^\mu j_\mu \equiv 0$.)

The present paper is now concerned with similar investigations of a further theory which has recently been constructed along the lines of those successful theories described above: the relativistic Schrödinger theory (RST) [2]. In order to see how the conservation laws emerge in this theory, let us start here with its central equation of motion which is the relativistic Schrödinger equation (RSE) for the N-component wavefunction ψ

$$\mathrm{i}\hbar c \mathcal{D}_\mu \psi = \mathcal{H}_\mu \cdot \psi. \tag{1.9}$$

The '*Hamiltonian*' $\mathcal{H}_\mu$ itself is a dynamical object (among others) obeying the field equation

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}_\mu + \frac{\mathrm{i}}{\hbar c}[\mathcal{H}_\mu, \mathcal{H}_\nu] = \mathrm{i}\hbar c \mathcal{F}_{\mu\nu}. \tag{1.10}$$

The internal consistency of the RST is expressed now by the fact that this '*integrability condition*' (1.10) implies both the Bianchi identity for $\mathcal{F}_{\mu\nu}$

$$\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} + \mathcal{D}_\mu \mathcal{F}_{\nu\lambda} + \mathcal{D}_\nu \mathcal{F}_{\lambda\mu} \equiv 0 \tag{1.11}$$

and the bundle identity for the wavefunction ψ

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu]\psi = \mathcal{F}_{\mu\nu} \cdot \psi. \tag{1.12}$$

But the crucial point here is that equation (1.10) (as only *one* part of the whole dynamical system) is already sufficient to establish additional conservation laws, apart from those following from the *complete* set of dynamical equations. Thus the RST is equipped with current densities of topological (h_μ) and dynamical (j_μ) origin, each type of current obeying the corresponding continuity equation:

$$\nabla^\mu j_\mu \equiv 0 \tag{1.13a}$$

$$\nabla^\mu h_\mu = \tfrac{3}{2}\epsilon^{\mu\nu\lambda\sigma}\,\mathrm{tr}(\mathcal{F}_{\sigma\mu} \cdot \mathcal{F}_{\nu\lambda}). \tag{1.13b}$$

But whereas the matter current (j_μ) naturally reflects the distribution of matter over spacetime, the topological current h_μ is built up exclusively by the Hamiltonian $\mathcal{H}_\mu$ and curvature $\mathcal{F}_{\mu\nu}$

$$h_\sigma = -2\mathrm{i}\epsilon^{\mu\nu\lambda}{}_\sigma \operatorname{tr}\left(\frac{\mathcal{H}_\mu \cdot \mathcal{H}_\nu \cdot \mathcal{H}_\lambda}{(\hbar c)^3} + \frac{3}{2}\frac{\mathcal{H}_\mu}{\hbar c} \cdot \mathcal{F}_{\nu\lambda}\right) \tag{1.14}$$

where both objects $\mathcal{H}_\mu$ and $\mathcal{F}_{\mu\nu}$ may be thought of as being rather independent of the actual matter distribution. Since the Hamiltonian is non-Hermitian ($\bar{\mathcal{H}}_\mu \neq \mathcal{H}_\mu$) in general, this current h_σ is complex

$$h_\sigma = t_\sigma + \mathrm{i}z_\sigma \tag{1.15}$$

and since its source (1.13*b*) is real, we must have

$$\nabla^\sigma t_\sigma = \tfrac{3}{2}\epsilon^{\mu\nu\lambda\sigma} \operatorname{tr}(\mathcal{F}_{\sigma\mu} \cdot \mathcal{F}_{\nu\lambda}) \tag{1.16a}$$

$$\nabla^\sigma z_\sigma \equiv 0. \tag{1.16b}$$

Thus both the matter current j_μ (1.13*a*) and the topological current z_μ (1.16*b*) have vanishing source and therefore define constants of motion, but of quite different nature: whereas the constant referring to the matter current j_μ fixes certain material properties of the physical system, the currents t_μ and z_μ eventually yield topological quantum numbers n_t, n_z which are closely related to the deRham cohomology group of the underlying spacetime:

$$48\pi^2 n_z = \oint_{C^3} {}^*\boldsymbol{z} \tag{1.17a}$$

$$48\pi^2 n_t = \oint_{C^3} {}^*\boldsymbol{t}. \tag{1.17b}$$

The integers n_z are then independent of how the dynamics is precisely realized. Obviously, this result is a further example of the occurrence of conservation laws through cooperation of the mathematical identities with the equations of motion (for the general significance of topological methods in gauge theories see also [3] and literature cited therein).

The present results are subsequently worked out in detail as follows.

In section 2, we first present a brief sketch of the basic structure of RST with emphasis on currents and conservation laws. From this approach to RST, the emergence of topological currents will be almost self-evident (section 3). The topological numbers n (1.17) are discussed together with the situation where the source of t_μ (1.16*a*) also vanishes without the curvature $\mathcal{F}_{\mu\nu}$ being zero. In section 4, a special example is presented in great detail in order to demonstrate the consistency of the general concepts and results. We consider here a special (Dirac) spinor field configuration over the Robertson–Walker universes and compute the corresponding topological numbers. Section 5 expresses the topological currents in terms of matter densities. In section 6 we present some arguments for why the quantum number due to ${}^*\boldsymbol{z}$ should always vanish for a SU(2) bundle, i.e. the Poincaré dual ${}^*\boldsymbol{z}$ of z_μ should always be an exact 3-form, in contrast to the case of ${}^*\boldsymbol{t}$. The discussion (section 7) uses these results in order to draw certain conclusions about the topology of solutions to the coupled Einstein–Yang–Mills–Higgs (EYMH) equations.

2. Matter currents

In order to elucidate the quite different nature of the matter and the topological currents, we first present a brief sketch of the general structure of RST with emphasis on the matter currents (and other matter densities).

Here, the basic concept is the intensity matrix $\mathcal{I}(x)$ which together with a certain Hermitian operator $\delta(x)$ $(=\bar{\delta})$ produces the associated extrinsic density $\Delta(x)$ in a gauge invariant way:

$$\Delta(x) = \mathrm{tr}\{\mathcal{I}(x) \cdot \delta(x)\}. \tag{2.1}$$

(In order for the (pseudo-) Hermiticity of $\mathcal{I}$ to be preserved during a change of gauge $\mathcal{I} \rightarrow \mathcal{S}^{-1} \cdot \mathcal{I} \cdot \mathcal{S}$, the gauge element $\mathcal{S}$ should be a member of one of the (pseudo-)unitary groups in N dimensions: $\bar{\mathcal{S}} = \mathcal{S}^{-1}$.)

In the present context, the most relevant object is the velocity operator v_μ generating the convection current j_μ

$$j_\mu = \mathrm{tr}(\mathcal{I} \cdot v_\mu) \tag{2.2}$$

which provides us with a first information of how the matter is distributed over spacetime. A special situation is encountered when matter is in a pure state $\psi(x)$, so that the intensity matrix $\mathcal{I}$ essentially adopts the specific form of a projector

$$\mathcal{I} \Rightarrow \psi \otimes \bar{\psi}. \tag{2.3}$$

This implies that the intensity matrix $\mathcal{I}(x)$ obeys the '*Fierz identity*' [4–6]

$$\mathcal{I}^2 = \rho \mathcal{I} \tag{2.4}$$

where the scalar density $\rho(x)$ is the simplest one of all the physical densities, namely

$$\rho \doteqdot \mathrm{tr}\,\mathcal{I}. \tag{2.5}$$

Clearly, for such a pure state the current density j_μ (2.2) is recast into the form

$$j_\mu = \bar{\psi} \cdot v_\mu \cdot \psi \tag{2.6}$$

or similarly for the scalar density ρ (2.5)

$$\rho = \bar{\psi} \cdot \psi \tag{2.7}$$

etc. The other physical densities of the theory are formed by the same procedure, for example the energy–momentum density $T_{\mu\nu}(x)$ is due to the corresponding operator $\mathcal{T}_{\mu\nu}$

$$T_{\mu\nu} = \mathrm{tr}(\mathcal{I} \cdot \mathcal{T}_{\mu\nu}) \tag{2.8}$$

but in the present paper our main interest lies upon the current densities j_μ (2.2).

In most situations, matter is not free but rather is acted upon by some force field $\mathcal{F}_{\mu\nu}$ (to be considered here as the Lie-algebra valued bundle curvature). Consequently, the question arises of how the densities are changing under the influence of that force field $\mathcal{F}_{\mu\nu}$, i.e. we have to look for the dynamical equations for those densities. Here, the crucial concept is the Hamiltonian $\mathcal{H}_\mu(x)$, a $\mathcal{GL}(N, \mathbb{C})$-valued 1-form acting upon the N-component bundle section $\psi(x)$ in the following way

$$\begin{gathered} \mathrm{i}\hbar c \mathcal{D}_\mu \psi = \mathcal{H}_\mu \cdot \psi \\ (\mathcal{D}_\mu \psi \doteqdot \partial_\mu \psi + \mathcal{A}_\mu \cdot \psi, \qquad \bar{\mathcal{A}}_\mu = -\mathcal{A}_\mu) \end{gathered} \tag{2.9}$$

(RSE). If matter is not in a pure state, the corresponding equation must refer to the intensity matrix $\mathcal{I}$ in place of ψ, i.e.

$$\mathcal{D}_\mu \mathcal{I} = \frac{\mathrm{i}}{\hbar c}[\mathcal{I} \cdot \bar{\mathcal{H}}_\mu - \mathcal{H}_\mu \cdot \mathcal{I}] \tag{2.10}$$

$$(\mathcal{D}_\mu \mathcal{I} \doteqdot \partial_\mu \mathcal{I} + [\mathcal{A}_\mu, \mathcal{I}]). \tag{2.11}$$

Obviously, the Hamiltonian governs the motion of matter but, on the other hand, it is generally thought that the origin of motion is the force field $\mathcal{F}_{\mu\nu}$. Therefore one expects a link between both objects $\mathcal{H}_\mu$ and $\mathcal{F}_{\mu\nu}$, and this is the '*integrability condition*'

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}_\mu + \frac{\mathrm{i}}{\hbar c}[\mathcal{H}_\mu, \mathcal{H}_\nu] = \mathrm{i}\hbar c \mathcal{F}_{\mu\nu}. \tag{2.12}$$

This equation is the first half of the equations of motion for $\mathcal{H}_\mu$ and simultaneously it ensures the existence of solutions $\psi(x)$ for the RSE (2.9) or its generalization (2.10). Observe also that the integrability condition (2.12) automatically implies the bundle identities for ψ

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu]\psi = \mathcal{F}_{\mu\nu} \cdot \psi \tag{2.13}$$

or for $\mathcal{I}$, respectively

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu]\mathcal{I} = [\mathcal{F}_{\mu\nu}, \mathcal{I}]. \tag{2.14}$$

Moreover, the integrability condition for the existence of solutions $\mathcal{H}_\mu(x)$ to equation (2.12) is just the well known Bianchi identity

$$\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} + \mathcal{D}_\mu \mathcal{F}_{\nu\lambda} + \mathcal{D}_\nu \mathcal{F}_{\lambda\mu} \equiv 0 \tag{2.15}$$

which identifies the force field $\mathcal{F}_{\mu\nu}$ as the curvature of the corresponding bundle connection $\mathcal{A}_\mu$:

$$\mathcal{F}_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{2.16}$$

But now comes the crucial point and this is the question of the conservation laws. Overwhelming experimental evidence tells us that elementary matter systems ('particles') carry along with them some physical invariants (rest mass, charge etc) which cannot be changed without destroying the particle. The mathematical expression of these conservation laws are the continuity equations

$$\nabla^\mu j_\mu = 0. \tag{2.17}$$

In the present context it is essential to remark that such a conservation law is *not* implied by those dynamical equations (2.9) or (2.10) considered up to now. Therefore it must be evident that the first field equation (2.12) for $\mathcal{H}_\mu$ is to be complemented by a second one which restricts the motion of matter just in such a way that the conservation laws (2.17) do actually hold! In order to find the corresponding restriction upon $\mathcal{H}_\mu$, one simply substitutes the current density j_μ (2.2) into the continuity requirement (2.17) and readily finds the following relationship between both objects v_μ and $\mathcal{H}_\mu$

$$\mathcal{D}^\mu v_\mu + \frac{\mathrm{i}}{\hbar c}[\bar{\mathcal{H}}^\mu \cdot v_\mu - v_\mu \cdot \mathcal{H}^\mu] = 2\mathcal{G}. \tag{2.18}$$

Here the new operator $\mathcal{G}$ ('convertor') must obey the following algebraic constraint

$$\mathcal{I} \cdot \mathcal{G} \equiv 0 \tag{2.19}$$

in order to close the matter system [2]. This condition can easily be obeyed over the whole spacetime by imposing upon the (Hermitian) convertor $\mathcal{G}$ the following equation of motion

$$\mathcal{D}_\mu \mathcal{G} = \frac{\mathrm{i}}{\hbar c}[\mathcal{G} \cdot \mathcal{H}_\mu - \bar{\mathcal{H}}_\mu \cdot \mathcal{G}] \tag{2.20}$$

which is slightly different from that for the intensity matrix $\mathcal{I}$ (2.10). But observe again that the bundle identity for $\mathcal{G}$

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu]\mathcal{G} = [\mathcal{F}_{\mu\nu}, \mathcal{G}] \tag{2.21}$$

is automatically obeyed by virtue of the integrability condition (2.12).

Summarizing the basic structure of RST we can say that the significance of the matter current j_μ lies in its ability to guarantee the existence of certain conservation laws (for closed systems) being expressed in differential terms by the continuity equation (2.17) (the gauge versions hereof are easily available, see [7, 8]). Thus the motion of matter under the action of the force $\mathcal{F}_{\mu\nu}$ and simultaneous regard of the conservation laws may be described by two steps.

(1) Determine the Hamiltonian $\mathcal{H}_\mu$, velocity operator v_μ and convertor $\mathcal{G}$ from their coupled field equations (2.12), (2.18), and (2.20).

(2) Determine the intensity matrix $\mathcal{I}$ from its field equation (2.10) with regard of the initial condition (2.19) (or determine the wavefunction ψ from the RSE (2.9), resp.).

As a result, one can then compute the densities $\Delta(x)$ (2.1) and thus one obtains all information about the physical system under consideration.

It should have become obvious now that the velocity operator v_μ represents one of the most significant quantities within RST, just on behalf of its close relationship with the conservation laws of the theory. Moreover, the velocity v_μ enters further important objects, for example the energy–momentum operator $\mathcal{T}_{\mu\nu}$ which may be defined as follows

$$\mathcal{T}_{\mu\nu} \doteqdot \tfrac{1}{2}\{v_\mu \cdot \mathcal{H}_\nu + \bar{\mathcal{H}}_\nu \cdot v_\mu\}. \tag{2.22}$$

In general this operator will be asymmetric (i.e. $\mathcal{T}_{\mu\nu} \neq \mathcal{T}_{\nu\mu}$) but its source just yields the expected Lorentz force and this result justifies interpreting the corresponding density $T_{\mu\nu}(x)$ (2.8) as the energy–momentum density of the system. In order to see this in more detail, compute the source of $T_{\mu\nu}$ with the help of equation (2.10) and find

$$\nabla^\mu T_{\mu\nu} = \mathrm{tr}\left\{\mathcal{I} \cdot \left(\mathcal{D}^\mu \mathcal{T}_{\mu\nu} + \frac{\mathrm{i}}{\hbar c}[\bar{\mathcal{H}}^\mu \cdot \mathcal{T}_{\mu\nu} - \mathcal{T}_{\mu\nu} \cdot \mathcal{H}^\mu]\right)\right\}. \tag{2.23}$$

Now substitute here the energy–momentum operator $\mathcal{T}_{\mu\nu}$ (2.22) and find by means of the '*conservation equation*' (2.18) together with the algebraic constraint (2.19) and the integrability condition (2.12)

$$\nabla^\mu T_{\mu\nu} = {}^{(L)}f_\nu \tag{2.24}$$

with the Lorentz force density ${}^{(L)}f_\nu$ being given by

$${}^{(L)}f_\nu = \mathrm{tr}(\mathcal{I} \cdot {}^{(L)}\mathcal{F}_\nu) \tag{2.25a}$$

$${}^{(L)}\mathcal{F}_\nu = \frac{\mathrm{i}\hbar c}{2}\{v^\mu \cdot \mathcal{F}_{\mu\nu} + \mathcal{F}_{\mu\nu} \cdot v^\mu\}. \tag{2.25b}$$

As expected, the Lorentz force is the product of velocity v_μ and field strength $\mathcal{F}_{\mu\nu}$, albeit in operator form. However, it should be remarked that the Lorentz result (2.25) could be attained only by imposing two further conditions upon the velocity operator v_μ and Hamiltonian $\mathcal{H}_\mu$, namely

$$\mathcal{D}_\mu v_\nu \equiv 0 \tag{2.26a}$$

$$\mathcal{D}_\mu(v^\nu \cdot \mathcal{H}_\nu + \bar{\mathcal{H}}_\nu \cdot v^\nu) \equiv 0. \tag{2.26b}$$

Combining these additional conditions with the conservation equation (2.18) we put

$$v^\mu \cdot \mathcal{H}_\mu = Mc^2\mathbb{1} + \mathrm{i}\hbar c\mathcal{G} \tag{2.27}$$

and this states that any solution $\psi(x)$ of the RSE (2.9) also satisfies the '*Dirac equation*' (put $v^\mu = \gamma^\mu$)

$$\mathrm{i}\hbar v^\mu \mathcal{D}_\mu \psi = Mc\psi. \tag{2.28}$$

Clearly, the choice (2.26*a*) of covariantly constant velocity operators v_μ implies further constraints to be satisfied. For instance, any operator v_μ must obey the bundle identity

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu] v_\lambda = [\mathcal{F}_{\mu\nu}, v_\lambda] - R^{\sigma}_{\lambda\mu\nu} v_\sigma \tag{2.29}$$

which then requires

$$[\mathcal{F}_{\mu\nu}, v_\lambda] \equiv R^{\sigma}_{\lambda\mu\nu} v_\sigma \tag{2.30}$$

where $R^{\sigma}_{\lambda\mu\nu}$ is the Riemannian curvature tensor of the underlying pseudo-Riemannian spacetime. However, implication (2.30) can easily be satisfied by identifying the velocity operators v_μ with the Dirac matrices γ_μ generating the Clifford algebra $\mathbb{C}(1, 3)$, i.e. we put

$$v_\mu \cdot v_\nu + v_\nu \cdot v_\mu = 2 g_{\mu\nu} \mathbb{1}. \tag{2.31}$$

As a consequence, the commutators of the velocity v_μ, being defined through

$$\Sigma_{\mu\nu} \doteqdot \tfrac{1}{4}[v_\mu, v_\nu] \tag{2.32}$$

obey the commutation relations

$$[\Sigma_{\mu\nu}, v_\lambda] = v_\mu g_{\lambda\nu} - v_\nu g_{\lambda\mu} \tag{2.33}$$

and therefore provide us with a basis of the Spin$(1, 3)$ algebra:

$$[\Sigma_{\mu\nu}, \Sigma_{\lambda\sigma}] = g_{\nu\lambda}\Sigma_{\mu\sigma} - g_{\nu\sigma}\Sigma_{\mu\lambda} + g_{\mu\lambda}\Sigma_{\sigma\nu} - g_{\mu\sigma}\Sigma_{\lambda\nu}. \tag{2.34}$$

Therefore we merely need to conceive the force $\mathcal{F}_{\mu\nu}$ to be the sum of its gravitational part ($R_{\sigma\lambda\mu\nu}$) and other gauge interactions $F_{a\mu\nu}$)

$$\begin{aligned} &\mathcal{F}_{\mu\nu} = \tfrac{1}{2} R_{\lambda\sigma\mu\nu}\Sigma^{\lambda\sigma} + F_{a\mu\nu}\tau^a \\ &([v_\mu, \tau^a] = 0) \end{aligned} \tag{2.35}$$

and then constraint (2.30) is automatically satisfied.

Although the preceding considerations show that the ordinary Dirac theory is *only one* special realization of the general RST, the significance of the velocity operators v_μ within the general framework of RST should have become clear now. Evidently, the existence of conservation laws is closely related to those matter currents j_μ induced by the velocity operator v_μ. But the striking point with RST is now that there are *additional* conservation laws which are not related to the matter currents but are of topological origin! How does this occur? Observe here the general structure of RST which is composed of two distinct building blocks:

(1) There is the level of *operators* (obeying the corresponding dynamical equations, e.g. (2.12), (2.18), or (2.20)).

(2) There is the level of *densities* (being constructed from the corresponding operators by means of the intensity matrix $\mathcal{I}$ or wavefunction ψ, e.g. (2.2), (2.5), or (2.8)).

The crucial point with this subdivision is now that the matter currents and associated conservation laws discussed so far are living on the second level, the level (ii) of densities, whereas the topological currents are already emerging on the first level (i) of operators which is surely the more fundamental one. This means that the physical densities do not influence the topological currents at all though these densities represent the specific spacetime distribution of matter.

Let us now see how the Hamiltonian $\mathcal{H}_\mu$ manages to directly build up some current (h_μ, say) without any reference to the densities.

3. Topological currents

The existence of such objects as the Hamiltonian $\mathcal{H}_\mu$ and curvature $\mathcal{F}_{\mu\nu}$, transforming homogeneously under a change of gauge $\mathcal{S}$

$$\mathcal{H}'_\mu = \mathcal{S}^{-1} \cdot \mathcal{H}_\mu \cdot \mathcal{S} \tag{3.1a}$$

$$\mathcal{F}'_{\mu\nu} = \mathcal{S}^{-1} \cdot \mathcal{F}_{\mu\nu} \cdot \mathcal{S} \tag{3.1b}$$

gives immediate rise to introduce a new kind of current density h_σ:

$$h_\sigma = -2\mathrm{i}\epsilon^{\mu\nu\lambda}{}_\sigma \, \mathrm{tr}\left(\frac{\mathcal{H}_\mu \cdot \mathcal{H}_\nu \cdot \mathcal{H}_\lambda}{(\hbar c)^3} + \frac{3}{2}\frac{\mathcal{H}_\mu}{\hbar c} \cdot \mathcal{F}_{\nu\lambda}\right). \tag{3.2}$$

In constrast to the Chern–Simons form [9] the present current h_σ is gauge invariant but it has the same source

$$\nabla^\sigma h_\sigma = \tfrac{3}{2}\epsilon^{\mu\nu\lambda\sigma}\, \mathrm{tr}(\mathcal{F}_{\sigma\mu} \cdot \mathcal{F}_{\nu\lambda}). \tag{3.3}$$

This is readily verified by *exclusive* use of the integrability condition (2.12), i.e. the conservation equation (2.18) is not needed here.

Observe that the source of the complex current h_σ is itself real (the curvature is taken to be anti-Hermitian, i.e. $\bar{\mathcal{F}}_{\mu\nu} = -\mathcal{F}_{\mu\nu}$). This implies that if we decompose both the new current and the Hamiltonian into its real and imaginary parts

$$h_\sigma = t_\sigma + \mathrm{i}z_\sigma \tag{3.4a}$$

$$\mathcal{H}_\mu = \hbar c(\mathcal{K}_\mu + \mathrm{i}\mathcal{L}_\mu) \tag{3.4b}$$

we find the real part of h_μ (3.2) being built up by the kinetic field $\mathcal{K}_\mu$ and the localization field $\mathcal{L}_\mu$ in the following way

$$t_\sigma = -2\mathrm{i}\epsilon^{\mu\nu\lambda}{}_\sigma\, \mathrm{tr}(\mathcal{K}_\mu \cdot \mathcal{K}_\nu \cdot \mathcal{K}_\lambda - 3\mathcal{L}_\mu \cdot \mathcal{L}_\nu \cdot \mathcal{K}_\lambda + \tfrac{3}{2}\mathcal{K}_\mu \cdot \mathcal{F}_{\nu\lambda}) \tag{3.5}$$

and analogously for the imaginary part z_σ:

$$z_\sigma = 2\mathrm{i}\epsilon^{\mu\nu\lambda}{}_\sigma\, \mathrm{tr}(\mathcal{L}_\mu \cdot \mathcal{L}_\nu \cdot \mathcal{L}_\lambda - 3\mathcal{L}_\mu \cdot \mathcal{K}_\nu \cdot \mathcal{K}_\lambda - \tfrac{3}{2}\mathcal{L}_\mu \cdot \mathcal{F}_{\nu\lambda}). \tag{3.6}$$

Clearly, the real source of h_μ must now be due to its real part t_σ, i.e.

$$\nabla^\sigma t_\sigma = \tfrac{3}{2}\epsilon^{\mu\nu\lambda\sigma}\, \mathrm{tr}(\mathcal{F}_{\sigma\mu} \cdot \mathcal{F}_{\nu\lambda}) \tag{3.7}$$

whereas the imaginary part z_σ is then necessarily found to be sourceless

$$\nabla^\sigma z_\sigma \equiv 0. \tag{3.8}$$

This may also be checked separately by splitting up the integrability condition (2.12) into its (anti-)Hermitian parts

$$\mathcal{D}_\mu\mathcal{K}_\nu - \mathcal{D}_\nu\mathcal{K}_\mu + \mathrm{i}[\mathcal{K}_\mu, \mathcal{K}_\nu] - \mathrm{i}[\mathcal{L}_\mu, \mathcal{L}_\nu] = \mathrm{i}\mathcal{F}_{\mu\nu} \tag{3.9a}$$

$$\mathcal{D}_\mu\mathcal{L}_\nu - \mathcal{D}_\nu\mathcal{L}_\mu + \mathrm{i}[\mathcal{K}_\mu, \mathcal{L}_\nu] - \mathrm{i}[\mathcal{K}_\nu, \mathcal{L}_\mu] = 0 \tag{3.9b}$$

and using these relations for differentiating both currents t_σ (3.5) and z_σ (3.6).

It is important to remark here that for the validity of the source relation (3.3), or equivalently for (3.7) and (3.8), nothing else is needed except the integrability condition (2.12). Especially, there is no need to use the conservation equation (2.18) (or its descendants) which was so crucial for those conservation laws based upon the matter currents j_μ (2.2). Furthermore, the new conservation law (3.8) does not either refer to the intensity matrix $\mathcal{I}$ but completely relies upon the use of the kinetic and localization fields and nothing else. Thus it becomes evident that the new continuity equation (3.8) is of quite a different nature than the former type (2.17). This readily becomes more evident by turning to the topological viewpoint. Defining the 3-form ${}^*\boldsymbol{z}$

$$ {}^*\boldsymbol{z} = \frac{1}{3!} z_{\mu\nu\lambda} \boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\lambda \tag{3.10} $$

as the (Poincaré) dual of the 1-form z_μ, i.e.

$$ \begin{aligned} z_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\sigma} z^\sigma \\ z_\lambda &= \frac{1}{3!} \epsilon_\lambda{}^{\mu\nu\sigma} z_{\mu\nu\sigma} \end{aligned} \tag{3.11} $$

the new continuity equation (3.8) reads

$$ \boldsymbol{d}^*\boldsymbol{z} = 0 \tag{3.12} $$

and this reveals the 3-form ${}^*\boldsymbol{z}$ as an element of deRham's cohomology algebra H^3 over spacetime [10]. The reason is that the corresponding cohomology class is actually independent of the kinetic field $\mathcal{K}_\mu$ and the curvature $\mathcal{F}_{\mu\nu}$. Therefore it is only the localization field $\mathcal{L}_\mu$ which determines that cohomology class. Thus, ${}^*\boldsymbol{z}$ becomes comparable with the concept of characteristic classes in differential topology [11].

In order to see this in more detail, we eliminate the curvature $\mathcal{F}_{\nu\lambda}$ from the topological current z_σ (3.6) in favour of the kinetic and localization fields by means of equation (3.9*a*) and find

$$ z_\sigma = 8\mathrm{i}\epsilon^{\mu\nu\lambda}{}_\sigma \operatorname{tr}(\mathcal{L}_\mu \cdot \mathcal{L}_\nu \cdot \mathcal{L}_\lambda) - 6\nabla_\nu[\epsilon_\sigma^{\mu\nu\lambda} \operatorname{tr}(\mathcal{L}_\mu \cdot \mathcal{K}_\lambda)]. \tag{3.13} $$

Or, in terms of the dual object $z_{\mu\nu\lambda}$ (3.11) this result reads

$$ z_{\mu\nu\lambda} = -24\mathrm{i} \operatorname{tr}[\mathcal{L}_\mu \cdot \mathcal{L}_\nu \cdot \mathcal{L}_\lambda - \mathcal{L}_\lambda \cdot \mathcal{L}_\nu \cdot \mathcal{L}_\mu] - 6[\nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\lambda\mu} + \nabla_\lambda f_{\mu\nu}] \tag{3.14} $$

where the 2-form $\boldsymbol{f} = \frac{1}{2} f_{\mu\nu} \boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu$ is composed of the kinetic and localization fields as follows

$$ f_{\mu\nu} = \operatorname{tr}[\mathcal{L}_\mu \cdot \mathcal{K}_\nu - \mathcal{L}_\nu \cdot \mathcal{K}_\mu]. \tag{3.15} $$

As a consequence, the essential part $\boldsymbol{\Lambda}$ of ${}^*\boldsymbol{z}$ is always independent of the curvature $\mathcal{F}_{\mu\nu}$, i.e. we put

$$ \begin{aligned} &{}^*\boldsymbol{z} = 8\boldsymbol{\Lambda} - 3\boldsymbol{d}\boldsymbol{f} \\ &(\boldsymbol{\Lambda} \doteqdot -\mathrm{i} \operatorname{tr}(\mathcal{L}_\mu \cdot \mathcal{L}_\nu \cdot \mathcal{L}_\lambda) \boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\lambda) \end{aligned} \tag{3.16} $$

and then the period n_z of ${}^*\boldsymbol{z}$ upon some 3-cycle C^3 of spacetime (with $\partial C^3 = \emptyset$) is influenced only by the localization field $\mathcal{L}_\mu$ alone:

$$ 48\pi^2 n_z = \oint_{C^3} {}^*\boldsymbol{z} = 8 \oint_{C^3} \boldsymbol{\Lambda}. \tag{3.17} $$

Observe here Stokes' theorem

$$ \oint_{C^3} \boldsymbol{d}\boldsymbol{f} = \oint_{\partial C^3} \boldsymbol{f} = 0. \tag{3.18} $$

Via integral (3.17), the localization field $\mathcal{L}_\mu$ associates a topological invariant n_z to the 3-cycle C^3 so that n_z is zero for those cases where the localization field commutes with itself:

$$ [\mathcal{L}_\mu, \mathcal{L}_\nu] = 0 \tag{3.19} $$

as was adopted in some preceding papers [12, 13]. Of course, non-trivial topological numbers (3.17) can arise only for spacetime manifolds whose Betti number b^3 (of third order) is non-zero: $b^3 \geqslant 1$ (Poincaré's lemma).

Finally, let us remark that the real part t_σ (3.5) of the complex current h_σ (3.2) is not so irrelevant as it may appear now through the preceding investigation of the topological current z_σ. In fact there are many instances, where the source of t_σ vanishes (cf (3.7)):

$$\epsilon^{\mu\nu\lambda\sigma}\,\mathrm{tr}(\mathcal{F}_{\sigma\mu}\cdot\mathcal{F}_{\nu\lambda})\equiv 0 \tag{3.20}$$

although the curvature $\mathcal{F}_{\mu\nu}$ is different from zero (see the example below). For such a situation the current t_σ also obeys a continuity equation

$$\nabla^\mu t_\mu \equiv 0 \tag{3.21}$$

which in turn yields a further topological number n_t via the analogue of equation (3.17)

$$48\pi^2 n_t = \oint_{C^3} {}^*\boldsymbol{t}. \tag{3.22}$$

In general, both periods n_z and n_t will be different from one another upon the same cycle C^3 and each of these numbers will adopt different values upon 3-cycles which are not homologous to each other.

But in any case, if the source equation (3.7) is rewritten in the language of forms

$$\begin{gathered}\boldsymbol{d}^*\boldsymbol{t} = (3!)\,\mathrm{tr}(\boldsymbol{\mathcal{F}}\wedge\boldsymbol{\mathcal{F}})\\ (\boldsymbol{\mathcal{F}} \doteqdot \tfrac{1}{2}\mathcal{F}_{\mu\nu}\boldsymbol{d}x^\mu\wedge\boldsymbol{d}x^\nu)\end{gathered} \tag{3.23}$$

we immediately see that the 4-form $\boldsymbol{f}$ $(=\boldsymbol{d}^*\boldsymbol{t})$ on the right of this equation is exact

$$\begin{gathered}\boldsymbol{dF}\equiv 0\\ (\boldsymbol{F} = (3!)\,\mathrm{tr}(\boldsymbol{\mathcal{F}}\wedge\boldsymbol{\mathcal{F}})).\end{gathered} \tag{3.24}$$

For the tangent bundle of our pseudo-Riemannian spacetime, this exact 4-form $\boldsymbol{F}$ $(= \frac{1}{4!}F_{\mu\nu\lambda\sigma}\boldsymbol{d}x^\mu\wedge\boldsymbol{d}x^\nu\wedge\boldsymbol{d}x^\lambda\wedge\boldsymbol{d}x^\sigma)$ is specialized into

$$F_{\mu\nu\lambda\sigma}\equiv(3!)\,\mathrm{tr}(\mathcal{F}_{[\mu\nu}\cdot\mathcal{F}_{\lambda\sigma]})\Rightarrow\frac{(3!)^2}{2}R_{\rho\kappa[\mu\nu}R_{\lambda\sigma]}{}^{\rho\kappa} \tag{3.25}$$

a result which has been known for a long time [14–16].

It may be instructive now to see how all these new concepts work in detail by considering a specific example.

4. Example: Dirac matter in Robertson–Walker universe

As mentioned above, RST as a quantum dynamical system is based upon the two fundamental concepts of *operators* and *densities*. The matter currents have been demonstrated to be special examples of densities but the two-level structure of RST has enabled us to construct topological currents also directly by exclusive use of the operators (i.e. with dispense of the intensity matrix $\mathcal{I}$). Furthermore, besides that two-level structure of the theoretical concepts, there also exists a similar subdivision for the quantum-dynamical equations: *integrability condition* (cf (2.12)) and *conservation equation* (cf (2.18)). This double duality (between operators and densities on the one hand and between ‘integrability’ and ‘conservation’ equations on the other hand) has some bearing upon the continuity equations obeyed by both types of currents. Whereas the matter currents rely mainly upon the conservation equation, the topological currents rely only upon the integrability condition. Surely, it would be very helpful now to consider a non-trivial example where both types of currents can be constructed explicitly and compared with one another.

4.1. Robertson–Walker universe

To this end, we consider the Dirac theory of spinning matter over a Robertson–Walker (RW) universe. The corresponding line element of the RW geometry is given by

$$\boldsymbol{d}s^2 = g_{\mu\nu}\boldsymbol{d}x^\mu \boldsymbol{d}x^\nu = \boldsymbol{d}\theta^2 - \mathcal{R}^2\boldsymbol{d}l^2 \tag{4.1}$$

where θ denotes the cosmic time, $\mathcal{R}$ is the scale parameter ('radius' of the universe), and $\boldsymbol{d}l$ is the spatial line element on the 3-surfaces of homogeneity and isotropy (θ = constant). Further geometric quantities are: the Hubble expansion rate H $(= \dot{\mathcal{R}}/\mathcal{R})$, the Hubble flow vector b_μ $(= \partial_\mu\theta)$ which is normalized to unity

$$b^\mu b_\mu = 1 \tag{4.2}$$

and its orthogonal projector $B_{\mu\nu}$ which simultaneously plays the part of the surface metric

$$g_{\mu\nu} = B_{\mu\nu} + b_\mu b_\nu \equiv B_{\alpha\mu}B^\alpha_\nu \tag{4.3a}$$

$$B_{\mu\nu}b^\nu = 0 \tag{4.3b}$$

$$B_{\mu\nu}B^\nu_\lambda = B_{\mu\lambda} \tag{4.3c}$$

$$B^\mu_\mu = 3. \tag{4.3d}$$

The Riemannian curvature tensor may now be expressed in terms of these objects as [17]

$$R_{\lambda\sigma\mu\nu} = \left(H^2 - \frac{\sigma}{\mathcal{R}^2}\right)[g_{\sigma\mu}g_{\lambda\nu} - g_{\sigma\nu}g_{\lambda\mu}] + \left(\dot{H} + \frac{\sigma}{\mathcal{R}^2}\right)[g_{\sigma\mu}b_\lambda b_\nu - g_{\sigma\nu}b_\lambda b_\mu + g_{\lambda\nu}b_\sigma b_\mu - g_{\lambda\mu}b_\sigma b_\nu] \tag{4.4}$$

where σ is the usual topological index ($\sigma = -1$: closed, $\sigma = 0$: flat, $\sigma = +1$: open universe).

Associated to the Levi–Civita connection $\boldsymbol{\Gamma}$ of the Riemannian metric $\boldsymbol{g}$ (4.3a) is the spin connection $\mathcal{A}$

$$\mathcal{A}_\mu = \tfrac{1}{2}A_{\alpha\beta\mu}\hat{\Sigma}^{\alpha\beta} = \tfrac{1}{2}A_{\lambda\sigma\mu}\Sigma^{\lambda\sigma} \tag{4.5}$$

$$(\hat{\Sigma}^{\alpha\beta} = B^\alpha_\mu B^\beta_\nu \Sigma^{\mu\nu}) \tag{4.6}$$

which takes its values in the Lie algebra of Spin$(1,3)$. The corresponding curvature $\mathcal{F}_{\mu\nu}$ of the RW universe is given by

$$\mathcal{F}_{\mu\nu} = \tfrac{1}{2}R_{\lambda\sigma\mu\nu}\Sigma^{\lambda\sigma} = \left(\frac{\sigma}{\mathcal{R}^2} - H^2\right)\Sigma_{\mu\nu} + \left(\frac{\sigma}{\mathcal{R}^2} + \dot{H}\right)b^\lambda[\Sigma_{\mu\lambda}b_\nu - \Sigma_{\nu\lambda}b_\mu]. \tag{4.7}$$

But for this specific curvature it is easy to prove by means of a little bit of Clifford algebra (cf also (3.25))

$$\mathrm{tr}(\mathcal{F}_{[\mu\nu}\cdot\mathcal{F}_{\lambda\sigma]}) \equiv 0 \tag{4.8}$$

and therefore the continuity equation (3.21) for the real counterpart t_σ (3.5) of the topological current z_σ will additionally hold.

4.2. Hamiltonian

Now that the relevant geometric preliminaries have been specified, one can next go on to determine the Hamiltonian $\mathcal{H}_\mu$ from the integrability condition (2.12). But for the sake of simplicity, we do not want to compute here the most general $\mathcal{H}_\mu$ for given RW-curvature $\mathcal{F}_{\mu\nu}$ (4.7) but we are satisfied with a subclass of solutions which reflect the RW-symmetry of the background geometry. Thus, our ansatz for the $\mathcal{GL}(4,\mathbb{C})$-valued Hamiltonian $\mathcal{H}_\mu$ ($\Rightarrow {}^{(a)}\mathcal{H}_\mu$) is

$$\frac{1}{\hbar c}{}^{(a)}\mathcal{H}_\mu = \{g\mathbb{1} + \tilde{g}\epsilon + G(b^\lambda\gamma_\lambda) + \tilde{G}(b^\lambda\tilde{\gamma}_\lambda)\}b_\mu - \left\{\left(W - \frac{m}{4}\right)\gamma^\lambda + \tilde{W}\tilde{\gamma}^\lambda\right\}B_{\lambda\mu} \\ -\mathrm{i}\{N\Sigma_{\mu\lambda} + \tilde{N}^*\Sigma_{\mu\lambda}\}b^\lambda. \tag{4.9}$$

Here, the ansatz parameters g, $\tilde{g}$, G, $\tilde{G}$, W, $\tilde{W}$, N, $\tilde{N}$ are complex homogeneous scalar fields over spacetime and therefore may be split up into real and imaginary parts as follows

$$\begin{aligned} g &= g_r + \mathrm{i}g_c \qquad & \tilde{g} &= \tilde{g}_r + \mathrm{i}\tilde{g}_c \\ G &= G_r + \mathrm{i}G_c \qquad & \tilde{G} &= \tilde{G}_r + \mathrm{i}\tilde{G}_c \\ W &= W_r + \mathrm{i}W_c \qquad & \tilde{W} &= \tilde{W}_r + \mathrm{i}\tilde{W}_c \\ N &= N_c - \mathrm{i}N_r \qquad & \tilde{N} &= \tilde{N}_c - \mathrm{i}\tilde{N}_r. \end{aligned} \tag{4.10}$$

Moreover, the $4 \times 4 = 16$ generators of the algebra $\mathcal{GL}(4,\mathbb{C})$ have been specified as $\mathbb{1}$, ϵ $(= \frac{1}{4!}\epsilon_{\mu\nu\lambda\sigma}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)$, γ^μ, $\tilde{\gamma}^\mu$ $(= \epsilon\gamma^\mu)$, and $\Sigma_{\mu\nu}$ $(= \frac{1}{4}[\gamma_\mu, \gamma_\nu])$. Observe here that the dual ${}^*\boldsymbol{\Sigma}$ of $\boldsymbol{\Sigma}$ is only a linear combination of the Spin(1, 3)-generators $\boldsymbol{\Sigma}$ but not a new element of the Clifford algebra $\mathbb{C}(1, 3)$:

$${}^*\Sigma_{\mu\nu} = -\epsilon\Sigma_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\sigma}\Sigma_{\lambda\sigma}. \tag{4.11}$$

Let us remark also that the present ansatz ${}^{(a)}\mathcal{H}_\mu$ (4.9) is a slight generalization of the former Hamiltonian ${}^{(s)}\mathcal{H}_\mu$

$$\frac{1}{\hbar c}{}^{(s)}\mathcal{H}_\mu = \frac{m}{4}\gamma_\mu + \frac{3}{2}\mathrm{i}b_\mu(N\mathbb{1} - \tilde{N}\epsilon) + (4b_\mu b_\lambda - g_{\mu\lambda})(W\mathbb{1} + \tilde{W}\epsilon)\gamma^\lambda \\ -\mathrm{i}b^\lambda(N\Sigma_{\mu\lambda} + \tilde{N}^*\Sigma_{\mu\lambda}) \tag{4.12}$$

which was used in a previous paper [13]. Evidently that special ${}^{(s)}\mathcal{H}_\mu$ (4.12) is recovered from the more general ${}^{(a)}\mathcal{H}_\mu$ (4.9) by putting

$$\begin{aligned} g &\Rightarrow \tfrac{3}{2}\mathrm{i}N \\ \tilde{g} &\Rightarrow -\tfrac{3}{2}\mathrm{i}\tilde{N} \\ G &\Rightarrow 3W + \frac{m}{4} \\ \tilde{G} &\Rightarrow 3\tilde{W}. \end{aligned} \tag{4.13}$$

The topological significance of this simplification process will readily become obvious.

4.3. Topological current

As soon as the Hamiltonian $\mathcal{H}_\mu$ is known, one can construct the complex current h_μ (3.2) in a purely algebraic manner irrespective of what kind of differential equation is obeyed by $\mathcal{H}_\mu$. Clearly, the source equation (3.3) is then satisfied only after one has additionally subjected the Hamiltonian to the integrability condition (2.12). Therefore let us first construct the current h_μ and discuss afterwards the implications of the integrability condition.

Introducing our special Hamiltonian ansatz ${}^{(a)}\mathcal{H}_\mu$ (4.9) into the general definition of the current h_μ (3.2) yields the following result (by means of a little bit of Clifford algebra)

$$h_\sigma \Rightarrow {}^{(a)}h_\sigma = \tilde{h}b_\sigma \tag{4.14}$$

with the complex homogeneous scalar field $\tilde{h}$ being given in terms of the ansatz scalars (4.10) as

$$\tilde{h} = 6\tilde{N}\left\{\tilde{N}^2 - 3N^2 - 12\tilde{W}^2 - 12\left(W - \frac{m}{4}\right)^2 + 3\left(H^2 - \frac{\sigma}{\mathcal{R}^2}\right)\right\}. \qquad (4.15)$$

Thus the complex current h_μ is found to be proportional to the Hubble flow b_μ and it vanishes whenever the ansatz parameter $\tilde{N}$ is zero. Moreover, the prefactor $\tilde{h}$ (4.15) is completely independent of the ansatz parameters g, $\tilde{g}$, G, $\tilde{G}$ and therefore the current h_μ is the same for a whole class of Hamiltonians. As we shall see below, this class character of h_μ will not be spoiled by incorporating the integrability condition (2.12) and this result renders the topological current z_σ comparable with the well known characteristic classes in differential topology which are constructed by means of the curvature $\mathcal{F}_{\mu\nu}$ in place of the Hamiltonian $\mathcal{H}_\mu$ [18, 19]. One could now decompose the present result ${}^{(a)}h_\mu$ (4.14) into the real and imaginary parts t_σ, z_σ (3.4*a*), but it is more convenient to first apply the integrability condition because this will yield a further simplification.

4.4. Integrability condition

In order for solutions $\psi(x)$ for the RSE (2.9) to actually exist, the Hamiltonian $\mathcal{H}_\mu$ must obey the integrability condition (2.12). Moreover, this condition also ensures the validity of the source equation for h_μ (3.3). But observe that this is *not sufficient* for the validity of the Dirac equation (2.28) which itself would then imply also the conservation law (2.17) for the matter current j_μ (2.6). However, the integrability condition (2.12) *is* sufficient to guarantee the topological conservation law (3.8). This assertion may be checked immediately by introducing our Hamiltonian ansatz ${}^{(a)}\mathcal{H}_\mu$ (4.9) into that integrability condition with the curvature $\mathcal{F}_{\mu\nu}$ being due to the RW case (4.7). With a little bit of Clifford algebra, this yields two kinds of equations for the four Hamiltonian coefficients W, $\tilde{W}$, N, $\tilde{N}$, namely their equations of motion and certain algebraic constraints among them. Here, the dynamical equations read

$$\dot{W} = -H\left(W - \frac{m}{4}\right) - (N+H)G + 2\mathrm{i}\tilde{g}\tilde{W} \qquad (4.16a)$$

$$\dot{\tilde{W}} = -H\tilde{W} - (N+H)\tilde{G} - 2\mathrm{i}\tilde{g}\left(W - \frac{m}{4}\right) \qquad (4.16b)$$

$$\dot{\tilde{N}} = -H\tilde{N} \qquad (4.16c)$$

$$\dot{N} + \dot{H} + \frac{\sigma}{\mathcal{R}^2} = (N+H)N + 4\left(W - \frac{m}{4}\right)^2 + 4\tilde{W}^2 + 4G\left(W - \frac{m}{4}\right) + 4\tilde{G}\tilde{W} - \tilde{N}^2. \qquad (4.16d)$$

Similarly, the algebraic constraints are found as

$$\tilde{N}\left(W - \frac{m}{4}\right) = 0 \qquad (4.17a)$$

$$\tilde{N}\tilde{W} = 0 \qquad (4.17b)$$

$$\tilde{N}(N+H) = 0 \qquad (4.17c)$$

$$\tilde{N}^2 = -\frac{\sigma}{\mathcal{R}^2} + (N+H)^2 + 4\left(W - \frac{m}{4}\right)^2 + 4\tilde{W}^2. \qquad (4.17d)$$

Let us first discuss the latter set of algebraic equations. If we want to have a non-trivial current ${}^{(a)}h_\mu$ (4.14), (4.15), then we obviously have to choose $\tilde{N} \neq 0$. But this then fixes the other ansatz scalars in (4.17) as

$$W = \frac{m}{4} \tag{4.18a}$$

$$\tilde{W} = 0 \tag{4.18b}$$

$$N = -H \tag{4.18c}$$

$$\tilde{N} = \pm\sqrt{-\sigma}\,\frac{1}{\mathcal{R}}. \tag{4.18d}$$

(Convince yourself that this special solution is actually admitted by the dynamical equations (4.16).) But with this result the complex current ${}^{(a)}h_\mu$ (4.14) becomes

$${}^{(a)}h_\mu \Rightarrow \mp 24\sigma\sqrt{-\sigma}\,\frac{1}{\mathcal{R}^3}b_\mu. \tag{4.19}$$

As a check it is easily verified here that the source equation for h_μ (3.3) is actually satisfied because: (i) for the present RW case the source of h_μ vanishes (cf (3.20)) and (ii) the derivative of the Hubble flow b_μ is given by

$$\nabla_\mu b_\nu = H B_{\nu\mu} \qquad (\Rightarrow \nabla^\mu b_\mu = 3H) \tag{4.20}$$

with H being just the Hubble expansion rate $(\dot{\mathcal{R}}/\mathcal{R})$.

Finally let us compute the topological quantum numbers n_z (3.17) and n_t (3.22). For a flat universe ($\sigma = 0$) the current ${}^{(a)}h_\mu$ (4.19) vanishes trivially. For an open universe ($\sigma = +1$), the real part t_σ (3.4*a*) vanishes and the topological current z_σ becomes

$${}^{(a)}z_\mu = \mp\frac{24}{\mathcal{R}^3}b_\mu \qquad (\sigma = +1). \tag{4.21}$$

However, since the open universe is equipped with the ordinary $\mathbb{R}^4$ topology, any two 3-cycles C^3 can be continuously deformed into one another without changing the topological number n_z. Thus, n_z must always be zero in open and flat universes (let one of the 3-cycles shrink to a point). This is equivalent to the fact that the (Poincaré) dual ${}^*\boldsymbol{z}$ (3.10) of the exact 1-form ${}^{(a)}z_\mu$ (4.21) is also exact.

But clearly for a closed universe ($\sigma = -1$) the dual ${}^*\boldsymbol{z}$ of the closed 1-form ${}^{(a)}z_\mu$ (4.21) would not be exact but here the imaginary part z_μ of ${}^{(a)}h_\mu$ (4.19) vanishes trivially and therefore we are again left with a vanishing quantum number n_z. However, for $\sigma = -1$ the *real* part ${}^{(a)}t_\mu$ becomes

$${}^{(a)}t_\mu = \pm\frac{24}{\mathcal{R}^3}b_\mu \qquad (\sigma = -1) \tag{4.22}$$

and thus is formally the same as the topological current (4.21) in the open universe. But in a *closed* universe the dual of the closed 1-form ${}^{(a)}t_\mu$ (4.22) is not exact and the corresponding topological number (3.22) becomes

$$48\pi^2 n_t = \oint_{C^3} {}^*\boldsymbol{t} = \oint_{\theta=\text{constant}} {}^{(a)}t_\mu\,\mathrm{d}S^\mu = \pm 48\pi^2 \qquad \left(\mathrm{d}S^\mu = b^\mu\,\mathrm{d}S, \oint_{\theta=\text{constant}} \mathrm{d}S = 2\pi^2\mathcal{R}^3\right) \tag{4.23}$$

which yields $n_t = \pm 1$.

This result simultaneously suggests that the topological number n_z (3.17) (and n_t (3.22) for the special case of vanishing source (3.20)) will always be found as an integer. Observe here that our original Hamiltonian ansatz ${}^{(a)}\mathcal{H}_\mu$ (4.9) contained eight complex parameters and one should have expected that these eight variables occur also in the complex current h_μ. Instead, the corresponding current ${}^{(a)}h_\mu$ (4.14) was built up by only four ansatz parameters $(W, \tilde{W}, N, \tilde{N})$. This means that a four-dimensional manifold of Hamiltonians ${}^{(a)}\mathcal{H}_\mu$ (being parametrized by $g, \tilde{g}, G, \tilde{G}$) collapses into a single current ${}^{(a)}h_\mu$. Further reduction of the field degrees of freedom occurred by applying the integrability condition (2.12) which left the *unique* current ${}^{(a)}h_\mu$ (4.19), which of course must then be a RW-symmetric 1-form. But it is just this collapse of a wide variety of Hamiltonians $\mathcal{H}_\mu$ into a single current h_μ which is to be expected when the topological numbers are emerging as integers. In this way the total set of Hamiltonians $\mathcal{H}_\mu$, as solutions of the integrability condition (2.12), will form discrete 'classes' being characterized by those topological numbers. In this sense, our Hamiltonian ${}^{(a)}\mathcal{H}_\mu$ (4.9) represents the RW-symmetric class which has topological numbers $n_t = 0, \pm 1$ and $n_z = 0$.

4.5. *Matter current*

Is there any relationship between the topological and matter currents? Especially, for the present RW-symmetric field configuration one would expect that the matter current j_μ (2.2) also adopts the corresponding RW-symmetric form and would then look just like the topological counterparts ${}^{(a)}z_\mu$ (4.21) or ${}^{(a)}t_\mu$ (4.22). One could think now that in order to check these suppositions one would first have to determine the intensity matrix $\mathcal{I}$ from its field equation (2.10) (resp. to determine the wavefunction ψ from the RSE (2.9)) before one can construct the matter current j_μ according to (2.2) (or according to (2.6), resp.). However, this would be a difficult task and it is better to resort to a simpler procedure.

In fact, if we want to get some information about the matter current j_μ we do not necessarily need to know its explicit form but we may be content to have its field equation. If the topological currents do not obey that field equation for the matter current, both types of current can never be identical (although anyone of them obeys the corresponding continuity equation). Therefore suppose for the moment that the matter current j_μ would also be of the RW-symmetric form, i.e.

$$\begin{aligned} &j_\mu = I b_\mu \\ &(I \doteq j^\mu b_\mu). \end{aligned} \tag{4.24}$$

Since homogeneity requires the scalar I to depend exclusively upon the cosmic time θ (remember $b_\mu = \partial_\mu \theta$) we find the following field equation for any RW-symmetric current (4.24) (cf (4.20)):

$$\begin{aligned} &\nabla_\mu j_\nu = \dot{I} b_\mu b_\nu + I H B_{\nu\mu} \\ &\left(\dot{I} \doteq \frac{\mathrm{d}I}{\mathrm{d}\theta}\right). \end{aligned} \tag{4.25}$$

On the other hand, the derivative of the matter current j_μ follows also by direct differentiation of its defining equation (2.2) (or (2.6), resp.) and by using the field equation for the intensity matrix $\mathcal{I}$ (2.10) (or using the RSE (2.9), resp.):

$$\nabla_\mu j_\nu = \frac{\mathrm{i}}{\hbar c} \operatorname{tr}(\mathcal{I} \cdot [\bar{\mathcal{H}}_\mu \cdot \gamma_\nu - \gamma_\nu \cdot \mathcal{H}_\mu]). \tag{4.26}$$

Introducing here our special Hamiltonian ansatz ${}^{(a)}\mathcal{H}_\mu$ (4.9) readily yields

$$\nabla_\mu j_\nu = 2\tilde{g}_r b_\mu \tilde{j}_\nu - 2\tilde{\rho}\tilde{G}_c b_\mu b_\nu + 8G_r b_\mu b^\lambda S_{\lambda\nu} + 2g_c b_\mu j_\nu + 2\rho G_c b_\mu b_\nu$$
$$-8\tilde{G}_r b_\mu b^{\lambda *} S_{\lambda\nu} + N_c[j_\mu b_\nu - I g_{\mu\nu}] + \tilde{N}_c j^\rho \epsilon_{\rho\mu\lambda\nu} b^\lambda. \tag{4.27}$$
$$\left(S_{\mu\nu} \doteq \frac{\mathrm{i}}{2}\,\mathrm{tr}\left(\mathcal{I}\cdot\Sigma_{\mu\nu}\right), {}^{*}S_{\mu\nu} = \frac{\mathrm{i}}{2}\,\mathrm{tr}(\mathcal{I}\cdot{}^{*}\Sigma_{\mu\nu})\right).$$

But this result is never symmetric ($\nabla_\mu j_\nu \neq \nabla_\nu j_\mu$) and therefore it cannot coincide with the previous supposition (4.24).

As a consequence, we conclude that in general the topological and matter currents will be quite different objects which are equipped with a rather different meaning within the framework of RST. Whereas the physical significance of the matter currents $\mathcal{J}_\nu$ as the sources for the gauge field $\mathcal{F}_{\mu\nu}$ is clear [7, 8], the physical role of the topological currents remains to be clarified.

5. Matter densities

Hitherto we have carefully kept the matter currents apart from the topological currents. Therefore it may appear now as a surprise that we go to establish an intimate link between the topological current z_μ and the intrinsic matter densities $\Delta_a(x)$

$$\Delta_a(x) = \mathrm{tr}(\mathcal{I}(x)\cdot\delta_a)$$
$$(a = 1\ldots N^2) \tag{5.1}$$

where δ_a ($= \bar{\delta}_a$) are N^2 fixed Hermitian operators over fibre space $\mathbb{C}^N$. Why does one want to have such a link between the topological quantities and the matter densities?

The motivation comes from some of the results of the preceding example. Remember here the fact that for the open universe ($\sigma = +1$) we got a non-trivial topological current ${}^{(a)}z_\mu$ (4.21) whose periods on any 3-cycle C^3, however, had to vanish. The reason for this is that the 3-distribution orthogonal to ${}^{(a)}z_\mu$ integrates to *open* 3-surfaces (θ = constant) which extend up to spatial infinity. Obviously, non-trivial periods of z_μ could have been obtained if the corresponding 3-surfaces would have been compact. But on the other hand, the closed universe ($\sigma = -1$) did not admit a non-trivial current z_μ, such as, for example ${}^{(a)}t_\mu$ (4.22) which implies some non-trivial quantum number n_t (4.23). These strange results find their natural explanation by evoking the matter densities $\Delta_a(x)$ (5.1) associated to the corresponding field configurations. Observe here that for a given configuration $\mathcal{I}(x)$ (or $\psi(x)$, resp.) the prescription (5.1) defines some embedding of spacetime $\{x\}$ into the density space $\{\Delta\}$: $x \mapsto \Delta(x)$ (more precisely we deal with a section $\mathcal{I}(x)$ of the bundle of densities over spacetime as the base space). But this density image of spacetime will in general carry a topology different from that of spacetime itself because any two events collapse into one single point in density space whenever they carry the same densities Δ. Considering now the pullback (with respect to the density map $x \rightarrow \Delta$) of some 3-form over density space $\{\Delta\}$ we immediately see that the integral surfaces of $\mathbf{\Lambda}$ do inherit their topology from that density image of spacetime. Thus the periods of $\mathbf{\Lambda}$ on the 3-cycles C^3 of spacetime reflect the density topology which in this way becomes the object of our interest.

Deferring some concrete example for this mechanism to the next section, we first have to say some words here about the density map $x \rightarrow \Delta$ and its pullback form $\mathcal{L}_\mu$.

In order to find the desired relationship between the intensity matrix $\mathcal{I}$ (as the collection of the intrinsic densities) and the localization field $\mathcal{L}_\mu$, we first change the connection $\mathcal{A}_\mu$

(2.16) into $\overset{\circ}{\mathcal{A}}_\mu$ by means of the kinetic field $\mathcal{K}_\mu$ (3.4*b*) according to

$$\overset{\circ}{\mathcal{A}}_\mu = \mathcal{A}_\mu + \mathrm{i}\mathcal{K}_\mu \tag{5.2}$$

and then we recast the equation of motion for $\mathcal{I}$ (2.10) into the following form

$$\begin{aligned} &\overset{\circ}{\mathcal{D}}_\mu\, \mathcal{I} = \mathcal{I}\cdot\mathcal{L}_\mu + \mathcal{L}_\mu\cdot\mathcal{I} \\ &(\overset{\circ}{\mathcal{D}}_\mu\, \mathcal{I} \doteq \partial_\mu\mathcal{I} + [\overset{\circ}{\mathcal{A}}_\mu, \mathcal{I}]). \end{aligned} \tag{5.3}$$

Consequently, any solution $\mathcal{I}(x)$ of (5.3) yields the desired density map $x \to \Delta$ and enables us to express the localization field $\mathcal{L}_\mu$ in terms of the matter densities.

The curvature $\overset{\circ}{\mathcal{F}}_{\mu\nu}$ of the new connection $\overset{\circ}{\mathcal{A}}_\mu$ has some interesting properties. First observe, that the new curvature $\overset{\circ}{\mathcal{F}}_{\mu\nu}$ is expressed in terms of the old $\mathcal{F}_{\mu\nu}$ (2.16) as

$$\overset{\circ}{\mathcal{F}}_{\mu\nu} = \mathcal{F}_{\mu\nu} + \mathrm{i}(\mathcal{D}_\mu\mathcal{K}_\nu - \mathcal{D}_\nu\mathcal{K}_\mu) - [\mathcal{K}_\mu, \mathcal{K}_\nu]. \tag{5.4}$$

Next remember the Hermitian part of the integrability condition (3.9*a*) and find

$$\overset{\circ}{\mathcal{F}}_{\mu\nu} = -[\mathcal{L}_\mu, \mathcal{L}_\nu]. \tag{5.5}$$

Thus, non-trivial topological numbers n_z (3.17) will emerge only for non-vanishing curvature $\overset{\circ}{\mathcal{F}}_{\mu\nu}$! Moreover, the anti-Hermitian part of the integrability condition (3.9*b*) may be reformulated as

$$\overset{\circ}{\mathcal{D}}_\mu\, \mathcal{L}_\nu - \overset{\circ}{\mathcal{D}}_\nu\, \mathcal{L}_\mu = 0 \tag{5.6}$$

so that the Bianchi identity for the new curvature $\overset{\circ}{\mathcal{F}}_{\mu\nu}$ is immediately seen to be valid

$$\overset{\circ}{\mathcal{D}}_\mu \overset{\circ}{\mathcal{F}}_{\nu\lambda} + \overset{\circ}{\mathcal{D}}_\nu \overset{\circ}{\mathcal{F}}_{\lambda\mu} + \overset{\circ}{\mathcal{D}}_\lambda \overset{\circ}{\mathcal{F}}_{\mu\nu} \equiv 0 \tag{5.7}$$

together with the closedness of the 3-form $\mathbf{\Lambda}$ (3.16). Finally let us remark also that the present results (5.3), (5.5) and (5.6) are consistent with the bundle identity for the intensity matrix $\mathcal{I}$, i.e.

$$[\overset{\circ}{\mathcal{D}}_\mu \overset{\circ}{\mathcal{D}}_\nu - \overset{\circ}{\mathcal{D}}_\nu \overset{\circ}{\mathcal{D}}_\mu]\mathcal{I} = [\overset{\circ}{\mathcal{F}}_{\mu\nu}, \mathcal{I}]. \tag{5.8}$$

After one is convinced now that the crucial relationship (5.3) between the matter and the topological objects is consistent, one can turn to the discussion of the corresponding solutions $\mathcal{I}(x)$ for a given connection $\overset{\circ}{\mathcal{A}}_\mu$ and localization field $\mathcal{L}_\mu$. Here it is interesting to observe that similarly the following types of equations also lead to the desired solutions:

$$\text{type I:} \quad \overset{\circ}{\mathcal{D}}_\mu\, \mathcal{L}_{\mathrm{I}} = \mathcal{L}_\mu\cdot\mathcal{L}_{\mathrm{I}} + \mathcal{L}_{\mathrm{I}}\cdot\mathcal{L}_\mu \tag{5.9a}$$

$$\text{type II:} \quad \overset{\circ}{\mathcal{D}}_\mu\, \mathcal{L}_{\mathrm{II}} = \mathcal{L}_\mu\cdot\mathcal{L}_{\mathrm{II}} - \mathcal{L}_{\mathrm{II}}\cdot\mathcal{L}_\mu \tag{5.9b}$$

$$\text{type III:} \quad \overset{\circ}{\mathcal{D}}_\mu\, \mathcal{L}_{\mathrm{III}} = -\mathcal{L}_\mu\cdot\mathcal{L}_{\mathrm{III}} + \mathcal{L}_{\mathrm{III}}\cdot\mathcal{L}_\mu \tag{5.9c}$$

$$\text{type IV:} \quad \overset{\circ}{\mathcal{D}}_\mu\, \mathcal{L}_{\mathrm{IV}} = -\mathcal{L}_\mu\cdot\mathcal{L}_{\mathrm{IV}} - \mathcal{L}_{\mathrm{IV}}\cdot\mathcal{L}_\mu. \tag{5.9d}$$

Clearly, the Hermitian part of any type-I solution is a possible intensity matrix, i.e.

$$\mathcal{I} \Rightarrow \mathcal{L}_{\mathrm{I}} + \bar{\mathcal{L}}_{\mathrm{I}}. \tag{5.10}$$

Similarly, the product of a type-I solution with type II or type III also yields solutions of the original equation (5.3)

$$\mathcal{I} \Rightarrow \mathcal{L}_{\mathrm{II}}\cdot\bar{\mathcal{L}}_{\mathrm{I}} + \mathcal{L}_{\mathrm{I}}\cdot\bar{\mathcal{L}}_{\mathrm{II}} \tag{5.11a}$$

$$\mathcal{I} \Rightarrow \bar{\mathcal{L}}_{\mathrm{III}}\cdot\mathcal{L}_{\mathrm{I}} + \bar{\mathcal{L}}_{\mathrm{I}}\cdot\mathcal{L}_{\mathrm{III}}. \tag{5.11b}$$

Or even combinations without presence of $\mathcal{L}_{\rm I}$ are possible, for example

$$\mathcal{I} \Rightarrow \bar{\mathcal{L}}_{\rm III} \cdot \bar{\mathcal{L}}_{\rm IV}^{-1} + \mathcal{L}_{\rm IV}^{-1} \cdot \mathcal{L}_{\rm III}. \tag{5.12}$$

Moreover, the intensity matrix $\mathcal{I}$ may be built up of products of higher orders.

A special case is encountered when the modified curvature $\overset{\circ}{\mathcal{F}}_{\mu\nu}$ (5.5) vanishes, i.e. when the localization fields are commuting (cf (3.19)). In this case, the matrix amplitude $\mathcal{L}$ may also be determined from the following equation

$$\overset{\circ}{\mathcal{D}}_\mu\, \mathcal{L} = \mathcal{L}_\mu \cdot \mathcal{L} \tag{5.13}$$

with the intensity matrix $\mathcal{I}$ being built up by $\mathcal{L}$ according to

$$\mathcal{I} \Rightarrow \mathcal{L} \cdot \bar{\mathcal{L}}. \tag{5.14}$$

This construction yields a non-negative matrix $\mathcal{I}$, provided the typical fibre $\mathbb{C}^N$ is equipped with a strictly Hermitian form (i.e. $\bar{\psi} \cdot \psi \geqslant 0$). However, within the present context of RST it seems unnecessary to exclude negative eigenvalues for $\mathcal{I}$ and therefore the more general forms (5.11), (5.12) for $\mathcal{I}$ may be admitted.

Finally, let us mention also the fact that the equation of motion for the convertor $\mathcal{G}$ (2.20) is rewritten in terms of the new connection $\overset{\circ}{\mathcal{A}}_\mu$ (5.2) as

$$\overset{\circ}{\mathcal{D}}_\mu\, \mathcal{G} = -\mathcal{G} \cdot \mathcal{L}_\mu - \mathcal{L}_\mu \cdot \mathcal{G} \tag{5.15}$$

i.e. the type-IV solution $\mathcal{L}_{\rm IV}(x)$ (5.9*d*) can be taken as the $\mathcal{G}$-field

$$\mathcal{G} \Rightarrow \mathcal{L}_{\rm IV} + \bar{\mathcal{L}}_{\rm IV}. \tag{5.16}$$

The generalizations hereof by building products again should be self-evident.

6. Example: Scalar Higgs doublet over RW universe

Recently [17], cosmological solutions for the coupled EYMH equations were presented but the results did not admit RW-symmetric solutions with non-vanishing topological number n_z over a closed universe [2]. In other words, one could not get solutions with non-commuting localization fields: $[\mathcal{L}_\mu, \mathcal{L}_\nu] \neq 0$ (see the remarks concerning equations (3.19) and (5.5)). As strange as this negative outcome may appear, it now finds its natural explanation within the present topological framework. In short, the point lies again in the fact, that the density image of spacetime is topologically trivial so that the pullback of the density map cannot generate closed 3-surfaces.

In order to see this effect in detail, let us first consider the general formalism for a SU(2) bundle and then apply the result to the present EYMH problem.

For fibre dimension $N = 2$, the (Hermitian) intensity matrix $\mathcal{I}$ can be expanded with respect to unity ($\mathbb{1}$) and the Pauli matrices (σ^j) according to

$$\mathcal{I} = \tfrac{1}{2}(\rho \mathbb{1} - s_j \sigma^j) \tag{6.1}$$

so that the intrinsic densities are given by

$$\rho = \operatorname{tr} \mathcal{I} \tag{6.2a}$$

$$s^j = g^{jk} s_k = \operatorname{tr}(\mathcal{I} \cdot \sigma^j) \tag{6.2b}$$

$$(g^{jk} = \operatorname{diag}(-1, -1, -1)). \tag{6.2c}$$

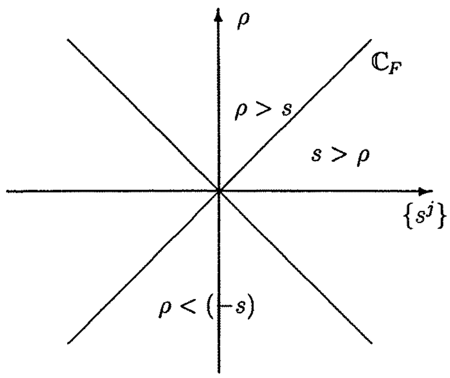

Figure 1. Density configuration space for $N = 2$. The Fierz cone $\mathbb{C}_F$ (6.3) cannot be traversed during the time evolution (2.10) of the intensity matrix $\mathcal{I}$ (6.1).

Thus, the density configuration space is a four-dimensional manifold parametrized by the 'coordinates' $\{\rho, s^j\}$, and the Fierz identity (2.4) reads

$$\rho^2 - s^2 = 0 \qquad (s^2 \doteq -s^j s_j) \tag{6.3}$$

which defines a 3-cone in this space ('*Fierz cone*' $\mathbb{C}_F$, figure 1). One can now easily show that with respect to the density map $x \mapsto \{\rho(x), s^j(x)\}$, given by some solution of (5.3), the spacetime manifold is mapped either into the interior ($\rho^2 - s^2 > 0$) or into the exterior ($\rho^2 - s^2 < 0$) of the Fierz cone, i.e. the density image $\{\rho(\theta), s^j(\theta)\}$ of any curve $x = x(\theta)$ of spacetime can never traverse the Fierz cone [8]. (This strongly reminds us of the analogous situation in special relativity where the light cone can never be traversed by a real particle world-line.) For the sake of simplicity, we restrict ourselves to the 'future' Fierz cone: $\rho > s$.

In the next step of our programme, we explicitly have to compute the pullback field $\mathcal{L}_\mu$ due to the density map. To this end we decompose the localization field $\mathcal{L}_\mu$ similarly as the intensity matrix $\mathcal{I}$ (6.1), i.e.

$$\mathcal{L}_\mu = L_\mu \mathbb{1} + L_{j\mu}\sigma^j. \tag{6.4}$$

Equation (5.3) then yields for the localization coefficients in terms of density derivatives

$$\partial_\mu \rho = 2(\rho L_\mu + s^j L_{j\mu}) \tag{6.5a}$$

$$\overset{\circ}{D}_\mu\, s_j = -2(\rho L_{j\mu} - s_j L_\mu). \tag{6.5b}$$

On the other hand, the desired 3-form $\mathbf{\Lambda}$ ($\doteq \frac{1}{3!}\Lambda_{\mu\nu\lambda}\, \boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\lambda$) in (3.16) is obtained as

$$\Lambda_{\mu\nu\lambda} = -12\varepsilon^{jkl} L_{j\mu} L_{k\nu} L_{l\lambda} \tag{6.6}$$

and this naturally suggests reformulating $\mathbf{\Lambda}$ in terms of the density derivatives (6.5) in order to identify the desired 3-distribution in density configuration space.

For that purpose it is very convenient to introduce the unit fibre element $\hat{s}_j$ according to

$$\hat{s}_j = s^{-1} s_j \qquad (\hat{s}^j \hat{s}_j = -1). \tag{6.7}$$

By means of this unit vector, one can split up all the SO(3) objects into their longitudinal and transverse parts, for example

$$L_{j\mu} = -\hat{s}_j(\hat{s}^j L_{j\mu}) + {}^{(\perp)}L_{j\mu} \doteq -\hat{s}_j \overset{\frown}{L}_\mu + {}^{(\perp)}L_{j\mu}. \tag{6.8}$$

For the derivative of the unit vector $\hat{s}_j$ (6.7) we then have

$$\overset{\circ}{D}_\mu \hat{s}_j = -2\frac{\rho}{s}{}^{(\perp)}L_{j\mu} \qquad (\rightsquigarrow \hat{s}^j \overset{\circ}{D}_\mu \hat{s}_j \equiv 0). \tag{6.9}$$

Furthermore, the unit section $\hat{s}_j$ can be used for the construction of an SO(2) reduction of the original SO(3) bundle. Thus, decomposing the curvature $\overset{\circ}{\mathcal{F}}_{\mu\nu}$ (5.5) with respect to the SU(2) generators τ^j

$$\overset{\circ}{\mathcal{F}}_{\mu\nu} = \overset{\circ}{F}_{j\mu\nu}\, \tau^j \qquad \left(\tau^j = -\frac{\mathrm{i}}{2}\sigma^j\right) \tag{6.10}$$

the corresponding curvature coefficients are found as

$$\overset{\circ}{F}_{j\mu\nu} = 4\varepsilon_j{}^{kl} L_{k\mu} L_{l\nu} \tag{6.11}$$

and may be used to construct the curvature $\hat{F}_{\mu\nu}$ of the reduced SO(2) subbundle as follows

$$\hat{F}_{\mu\nu} = \hat{s}^j\, \overset{\circ}{F}_{j\mu\nu} + \varepsilon^{jkl}\hat{s}_j(\overset{\circ}{D}_\mu\, \hat{s}_k)(\overset{\circ}{D}_\nu\, \hat{s}_l). \tag{6.12}$$

As a check, the SO(2) curvature again must obey the Bianchi identity

$$\nabla_\lambda \hat{F}_{\mu\nu} + \nabla_\mu \hat{F}_{\nu\lambda} + \nabla_\nu \hat{F}_{\lambda\mu} \equiv 0 \tag{6.13}$$

which is, however, easily verified by evoking the SO(3) bundle identity

$$[\overset{\circ}{D}_\mu \overset{\circ}{D}_\nu - \overset{\circ}{D}_\nu \overset{\circ}{D}_\mu]\hat{s}_j \equiv \varepsilon_j{}^{kl}\, \overset{\circ}{F}_{k\mu\nu}\, \hat{s}_l. \tag{6.14}$$

For our present context, the topological significance of the reduced bundle curvature $\hat{\boldsymbol{F}} = \frac{1}{2}\hat{F}_{\mu\nu}\boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu$ lies in the fact that it simultaneously represents a characteristic class of that SO(2) bundle: the Euler class [10]. Consequently, if we consider some 2-cycle C^2 of spacetime which encloses the world-line of an isolated zero ($s^j = 0$) of the section $s^j(x)$ we obtain the Poincaré–Hopf index n_H of that zero by means of the period of $\hat{\boldsymbol{F}}$ upon C^2:

$$n_H = \oint_{C^2} \hat{\boldsymbol{F}} \tag{6.15}$$

(figure 2). If the 2-cycle C^2 encloses more than one world-line, integral (6.15) yields the sum of the individual Hopf indices. For our present SO(3) bundle the Hopf index of the zeros of the section $s^j(x)$ is quantized in units of 4π: $n_H = 4\pi n$, $n = 0, \pm 1, \pm 2, \ldots$.

The point with the Euler class $\hat{\boldsymbol{F}}$ of the subbundle is now that the desired 3-form $\boldsymbol{\Lambda}$ (6.6) can be expressed in terms of this object: one merely applies the splitting (6.8) into the transverse and longitudinal parts, eliminates the transverse part ${}^{(\perp)}L_{j\mu}$ in favour of the density derivative (6.9) and finds

$$\varepsilon^{jkl} L_{j\mu} L_{k\nu} L_{l\lambda} = -\frac{1}{4}\frac{s^2}{\rho^2 - s^2}\{\overset{\frown}{L}_\mu\, \hat{F}_{\nu\lambda} + \overset{\frown}{L}_\nu\, \hat{F}_{\lambda\mu} + \overset{\frown}{L}_\lambda\, \hat{F}_{\mu\nu}\}. \tag{6.16}$$

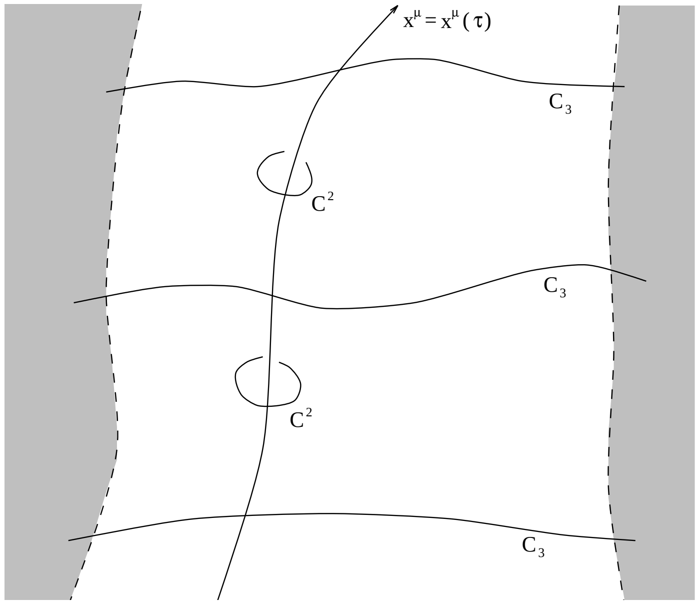

Figure 2. Poincaré–Hopf index and boundary conditions. The closed 2-surface C^2 surrounds the world-line $x^\mu = x^\mu(\tau)$ of some isolated zero of the section $s^j(x)$ (τ: proper length of the world-line). The Hopf index n_H (6.15) remains invariant through shifting C^2 along the world-line. Non-trivial values (6.21) of the topological number n_z are obtained when the projective coordinate s/ρ tends to a constant at spatial infinity (shaded area).

Furthermore, the longitudinal part $\overset{\frown}{L}_\mu$ (6.8) still occurring here is expressed in terms of the density derivatives as

$$\overset{\frown}{L}_\mu = \frac{1}{2}\frac{1}{1-\left(\frac{s}{\rho}\right)^2}\partial_\mu\left(\frac{s}{\rho}\right). \tag{6.17}$$

Therefore we can finally eliminate the longitudinal part $\overset{\frown}{L}_\mu$ from result (6.16) in favour of the density derivatives and then finally end up with the desired 3-form $\boldsymbol{\Lambda}$ as

$$\Lambda_{\mu\nu\lambda} = 3\{a_\mu \hat{F}_{\nu\lambda} + a_\nu \hat{F}_{\lambda\mu} + a_\lambda \hat{F}_{\mu\nu}\} \tag{6.18}$$

with the gradient field $a_\mu \equiv \partial_\mu a$ being determined by the scalar function a through

$$a = \frac{1}{4}\left(\frac{\frac{s}{\rho}}{1-\left(\frac{s}{\rho}\right)^2} - \frac{1}{2}\ln\frac{1+\frac{s}{\rho}}{1-\frac{s}{\rho}}\right). \tag{6.19}$$

Thus the result for the desired 3-form $\boldsymbol{\Lambda}$ (3.16) finally becomes

$$\boldsymbol{\Lambda} = 3(\boldsymbol{d}a)\wedge\hat{\boldsymbol{F}} = 3\boldsymbol{d}(a\hat{\boldsymbol{F}}) \tag{6.20}$$

and $\boldsymbol{\Lambda}$ is revealed for the present case of fibre dimensions $N = 2$ as an *exact* 3-form. As a consequence, the topological number n_z (3.17) must always be zero for the present case ($N = 2$). Evidently, the reason for this result is that $\boldsymbol{\Lambda}$ is the pullback (from density space to spacetime) of some 3-form over the topologically modified density configuration space. Since the function a (6.19) exclusively depends upon the ratio s/ρ, this modification consists of converting the interior ($\rho > s$) of the 'future' Fierz cone (6.3) into some non-compact submanifold of projective 3-space equipped with the 'volume' 3-form $\mathbf{\Lambda}$ (6.20). Naturally, the integral surfaces of the pullback of $\mathbf{\Lambda}$ must then also be non-compact 3-surfaces (e.g. extending to spatial infinity) which implies the vanishing of the topological number n_z (3.17). Non-trivial values may only be obtained if integral (3.17) is taken over some non-compact 3-surface C_3 extending to spatial infinity where the boundary condition $a \to a_\infty = a(s_\infty/\rho_\infty)$ is to be imposed (figure 2):

$$\int_{C_3} \mathbf{\Lambda} \Rightarrow \oint_{C^2} \hat{\boldsymbol{F}} \cdot \int_0^{a_\infty} \boldsymbol{d}a = 4\pi n a_\infty. \tag{6.21}$$

7. Einstein–Yang–Mills–Higgs system

As a concrete demonstration of the preceding general results concerning SU(2) bundles, we consider now the RW-symmetric solutions of the coupled EYMH equations [2, 17]. Especially, we want to inspect whether the topological number n_z (3.17) really vanishes on account of the non-compactness of the corresponding pullback 3-surfaces (cf the remarks at the beginning of section 5).

The Yang–Mills part of the total EYMH system reads†

$$\begin{aligned} &\mathcal{D}^\mu \mathcal{F}_{\mu\nu} = 4\pi\alpha \mathcal{J}_\nu \\ &\left(\alpha = \frac{g^2}{\hbar c} \ldots \text{'fine-structure constant'}\right) \end{aligned} \tag{7.1}$$

or after decomposition with respect to some suitable basis $\{\tau^j\}$ of the SU(2) algebra ($\mathcal{J}_\nu = j_{k\nu}\tau^k$, etc)

$$\begin{aligned} &D^\mu F_{k\mu\nu} = 4\pi\alpha j_{k\nu} \\ &(D_\mu j_{k\nu} \doteq \nabla_\mu j_{k\nu} + \varepsilon_k{}^{jl} A_{j\mu} j_{l\nu}). \end{aligned} \tag{7.2}$$

Now the only possible form for the curvature $\mathcal{F}_{\mu\nu}$, being consistent with the RW-symmetry, reads

$$\mathcal{F}_{\mu\nu} = -\frac{1}{4} f_\parallel [\mathcal{B}_\mu, \mathcal{B}_\nu] + \frac{\mathrm{i}}{2} f_\perp [b_\mu \mathcal{B}_\nu - b_\nu \mathcal{B}_\mu] \tag{7.3}$$

where b_μ is again the Hubble flow (4.2) and $\mathcal{B}_\mu$ ($\doteq B_{j\mu}\sigma^j$) denotes the $\mathcal{SU}(2)$-valued space part of the tetrad $\{B_{\alpha\mu}\}$, cf (4.3a). According to the RW-symmetry, the coefficients $f_\parallel$ and $f_\perp$ are homogeneous scalar fields (e.g. $f_\parallel = f_\parallel(\theta)$) which must obey the Bianchi constraint (cf (1.11))

$$\dot{f}_\parallel + 2H f_\parallel - 2\frac{\zeta}{\mathcal{R}} f_\perp \equiv 0 \tag{7.4}$$

where $\zeta = \zeta(\theta)$ is another homogeneous scalar. Thus we may put

$$f_\parallel = \frac{\sigma + \zeta^2}{\mathcal{R}^2} \tag{7.5a}$$

$$f_\perp = \frac{\dot{\zeta}}{\mathcal{R}} \tag{7.5b}$$

† Since the coupling constant g has been absorbed here into the physical field strength ${}^{(ph)}\mathcal{F}_{\mu\nu}$ in order to obtain the geometric notion of curvature $\mathcal{F}_{\mu\nu}$ ($\doteq \frac{g}{\hbar c}\,{}^{(ph)}\mathcal{F}_{\mu\nu}$), the generalized Maxwell equation (7.1) must contain the fine structure constant α as a prefactor of the matter current $\mathcal{J}_\mu$.

Table 1. Type of universe and associated Hamiltonian $\mathcal{H}_\mu$ (7.7).

	scalar function h	
σ	$h_r =$	$h_c =$
0 (flat)	$-\frac{\zeta}{2\mathcal{R}}$	0
-1 (closed)	$\frac{\pm 1-\zeta}{2\mathcal{R}}$	0
$+1$ (open)	$-\frac{\zeta}{2\mathcal{R}}$	$\pm\frac{1}{2\mathcal{R}}$

and the Bianchi identity (7.4) is safely satisfied, with the (gauge plus coordinate covariant) derivative of the triad field $\mathcal{B}_\mu$ being given by [17]

$$\mathcal{D}_\mu \mathcal{B}_\nu = -Hb_\nu \mathcal{B}_\mu + \frac{\mathrm{i}}{2}\frac{\zeta}{\mathcal{R}}[\mathcal{B}_\mu, \mathcal{B}_\nu]. \tag{7.6}$$

Once the SU(2)-curvature has been specified consistently through equation (7.3), the next step again consists of determining the Hamiltonian $\mathcal{H}_\mu$ from the integrability condition (2.12). Since $\mathcal{H}_\mu$ is some non-Hermitian 1-form, taking its values in the Lie algebra $\mathcal{GL}(2, \mathbb{C})$, we try the following ansatz

$$\mathcal{H}_\mu = \eta b_\mu \mathbb{1} + h\mathcal{B}_\mu \tag{7.7}$$

with the two complex scalar fields η, h

$$\eta = \eta_r + \mathrm{i}\eta_c \tag{7.8a}$$

$$h = h_r + \mathrm{i}h_c \tag{7.8b}$$

still to be determined. The corresponding solutions are collected in table 1.

But with the Hamiltonian being known, one readily computes the complex current h_μ (3.2) as

$$h_\mu = 6h(3f_\| - 4h^2)b_\mu \tag{7.9}$$

which is the SU(2) analogue of the Spin(1, 3) result (4.14); (4.15). Furthermore, decomposing this current into its real and imaginary parts according to (3.4a) readily yields

$$t_\mu = 6h_r\left(3\frac{\sigma+\zeta^2}{\mathcal{R}^2} + 12h_c^2 - 4h_r^2\right)b_\mu \tag{7.10a}$$

$$z_\mu = 6h_c\left(4h_c^2 - 12h_r^2 + 3\frac{\sigma+\zeta^2}{\mathcal{R}^2}\right)b_\mu. \tag{7.10b}$$

Therefore, one merely needs to take the scalar h from table 1 and introduce it into the present result (7.9) to find for the topological current

$$z_\mu = \begin{cases} 0 & \sigma = 0, -1 \\ \pm\frac{12}{\mathcal{R}^3}b_\mu & \sigma = +1. \end{cases} \tag{7.11}$$

Thus, just as for the Spin(1, 3) case (4.22), the present topological current z_μ (7.11) is non-trivial only for the *open* universe ($\sigma = +1$) and includes here as integral surfaces the open time-slices $\theta =$ constant, which extend to spatial infinity. This outcome is in complete agreement with the results of the preceeding section and thus we have vanishing topological number $n_z = 0$ for all kinds of universes ($\sigma = 0, \pm 1$). However, it should be remarked here that the Spin(1, 3) result (4.22) does refer to a very special field configuration and we cannot be sure whether there are also configurations with non-trivial values of the topological number n_z. On the other hand, the present results for SU(2) bundles are quite general so that we can rely upon the fact that n_z must always be zero (apart from those pathological cases (6.21)).

Finally, let us also briefly consider the real part t_μ (7.10*a*) which is found by means of table 1 as

$$t_\mu = \begin{cases} -6\dfrac{\zeta^3}{\mathcal{R}^3}b_\mu & \sigma = 0 \\ 6\dfrac{-\zeta^3 + 3\zeta \mp 2}{\mathcal{R}^3}b_\mu & \sigma = -1 \\ -6\dfrac{\zeta^3 + 3\zeta}{\mathcal{R}^3}b_\mu & \sigma = +1. \end{cases} \tag{7.12}$$

For a discussion of this result, first remember the corresponding Yang–Mills field $\mathcal{F}_{\mu\nu}$ (7.3). Obviously, it is possible to have a Yang–Mills vacuum ($\mathcal{F}_{\mu\nu} = 0$) only for a closed ($\sigma = -1$) and for a flat universe ($\sigma = 0$). The associated vacuum values are $\zeta = 0$ (for $\sigma = 0$) and $\zeta = \pm 1$ (for $\sigma = -1$). Thus, a topologically non-trivial situation is encountered only for the closed universe ($\sigma = -1$) where the two different vacuum phases ($\zeta = \pm 1$) are equipped with two *different* topological numbers n_t (3.22), namely n_t may be 0 or 1 (resp. $n_t = 0, -1$).

Consequently, we can have field configurations which initially ($\theta_{\text{in}} \to -\infty$) are in one kind of the Yang–Mills vacuum (e.g. $n_{t,\text{in}} = 0$) and finally ($\theta_{\text{fin}} \to +\infty$) terminate at the other kind of vacuum (i.e. $n_{t,\text{fin}} = +1$). In Euclidean field theory, such a vacuum transition has been called an '*instanton*' [20, 21]. Our present result now states that those (real) instanton solutions are possible also over a *pseudo*-Riemannian spacetime. The essential point here is that the radius $\mathcal{R}$ (4.1) of the universe may change with respect to cosmic time θ and this expansion just acts as a kind of damping effect which is necessary for settling down the Yang–Mills variable ζ to one of its vacuum values (± 1).

The free ($\mathcal{J}_\nu = 0$) Yang–Mills equation (7.1) reads for the variable ζ [17]

$$\ddot{\zeta} + H\dot{\zeta} + 2\frac{\sigma + \zeta^2}{\mathcal{R}^2}\zeta = 0 \tag{7.13}$$

which may be re-interpreted as the Newtonian equation for the motion of some point-particle in a double-well potential ($\sigma = -1$). As usual, a first integral of the mechanical equation of motion (7.13) is the well known energy conservation law

$$\frac{1}{2}\dot{\zeta}^2 + \frac{1}{2}\frac{(\sigma + \zeta^2)^2}{\mathcal{R}^2} = \frac{4\pi}{3}\frac{W_*}{\hbar c}\left(\frac{\mathcal{R}_*^2}{\mathcal{R}}\right)^2 \tag{7.14}$$
$$(\mathcal{R}_*, W_* = \text{constant}).$$

One may now imagine a cosmological situation where initially ($\theta_{\text{in}} \to -\infty$) the radius $\mathcal{R}$ is a large constant ($\rightsquigarrow H = 0$) and the Yang–Mills field is in one of its vacuum phases ($\zeta \approx -1$, $n_t = 0$). When the universe begins to contract, the Hubble expansion rate becomes *negative* and excites the variable ζ to oscillate with ever increasing amplitude around its original vacuum value ($\zeta = -1$). Thereby the Newtonian particle may slide into the other well ($\zeta \approx +1$) and if the universe re-expands in a suitably way ($\rightsquigarrow H > 0$), the particle may settle down at this new equilibrium value ($\zeta = +1$). In field-theoretic terms this picture states that the Yang–Mills field has been cast out of its original vacuum value ($\zeta = -1$) by *contraction* of the universe ($H < 0$) and has then been forced down into its new vacuum value ($\zeta = +1$) by *expansion* of the universe ($H > 0$).

8. Discussion

Since RST is a rather general framework which also embraces the spinor as the scalar realizations (Dirac and Klein–Gordon theories), the general results of RST must be valid for all its specific realizations. Thus the topological conservation laws, found in this paper, must hold for any possible realization of RST, especially for the Dirac case which was exemplified explicitly by the preceding considerations. In this way, RST reveals its usefulness as a special technique for investigating the well known wave equations of relativistic quantum mechanics. Indeed, one may doubt whether the present topological conservation laws could have been discovered also via the traditional approach to relativistic quantum mechanics. But clearly, whenever a formal method works very well, one is tempted to think that it will also reflect what is really going on in nature. In this sense, the RST has already been used to account for the non-local phenomena in quantum theory [22, 23]. It remains to be shown whether a reformulation of the whole quantum theory within the general framework of RST actually yields a better understanding of the microworld.

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